

GENERATION OF SUBORDINATED HOLOMORPHIC SEMIGROUPS VIA YOSIDA'S THEOREM

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*To Charles Batty, colleague and friend, on the occasion
of his sixtieth anniversary with admiration*

ABSTRACT. Using functional calculi theory, we obtain several estimates for $\|\psi(A)g(A)\|$, where ψ is a Bernstein function, g is a bounded completely monotone function and $-A$ is the generator of a holomorphic C_0 -semigroup on a Banach space, bounded on $[0, \infty)$. Such estimates are of value, in particular, in approximation theory of operator semigroups. As a corollary, we obtain a new proof of the fact that $-\psi(A)$ generates a holomorphic semigroup whenever $-A$ does, established recently in [8] by a different approach.

1. INTRODUCTION

Bernstein functions play an important role in analysis, and in particular, in the study of Lévy processes in probability theory. Recently they found a number of applications in operator and ergodic theories, mainly in issues related to rates of convergence of semigroups and related operator families. At a core of many applications of Bernstein functions is an abstract subordination principle going back to Bochner, Nelson and Phillips (see [19, p. 171] for more on its historical background). Given a Bernstein function ψ and a generator $-A$ of a bounded C_0 -semigroup on a Banach space X , the principle allows one to define the operator $-\psi(A)$ which again is the generator of a bounded C_0 -semigroup on X . Thus, it is natural to ask whether Bernstein functions preserve other classes of (bounded) semigroups relevant for applications such as holomorphic, differentiable or any of their subclasses. This paper treats the permanence of the class of holomorphic semigroups under Bernstein functions.

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Recall that a positive function $g \in C^\infty(0, \infty)$ is called *completely monotone* if

$$(-1)^n g^{(n)}(\tau) \geq 0, \quad \tau > 0,$$

for each $n \in \mathbb{N}$.

A positive function $\psi \in C^\infty(0, \infty)$ is called a *Bernstein function* if its derivative is completely monotone.

A basic property of Bernstein functions is that their exponentials arise as Laplace transforms of a uniquely defined convolution semigroups of subprobability measures. This property is a core of the notion of subordination discussed below.

Recall that a family of Radon measures $(\mu_t)_{t \geq 0}$ on $[0, \infty)$ is called a vaguely continuous convolution semigroup of subprobability measures if for all $t \geq 0, s \geq 0$,

$$\mu_t([0, \infty)) \leq 1, \quad \mu_{t+s} = \mu_t * \mu_s, \quad \text{and} \quad \text{vague} - \lim_{t \rightarrow 0+} \mu_t = \delta_0,$$

where δ_0 stands for the Dirac measure at zero. Such a semigroup is often called a subordinator. The next classical characterization of Bernstein functions goes back to Bochner and can be found e.g. in [19, Theorem 5.2].

Theorem 1.1. *A function $\psi : (0, \infty) \rightarrow (0, \infty)$ is Bernstein if and only if there exists a vaguely continuous convolution semigroup $(\mu_t)_{t \geq 0}$ of subprobability measures on $[0, \infty)$ such that*

$$(1.1) \quad \hat{\mu}_t(\tau) := \int_0^\infty e^{-s\tau} \mu_t(ds) = e^{-t\psi(\tau)}, \quad \tau > 0,$$

for all $t \geq 0$.

Theorem 1.1 has its operator-theoretical counterpart. One of most natural ways to construct a new C_0 -semigroup from a given one is to use subordinators. Recall that if $(e^{-tA})_{t \geq 0}$ is a bounded C_0 -semigroup on a Banach space X and $(\mu_t)_{t \geq 0}$ is a vaguely continuous convolution semigroup of bounded Radon measures on $[0, \infty)$ then the formula

$$(1.2) \quad e^{-tA} = \int_0^\infty e^{-sA} \mu_t(ds), \quad t \geq 0,$$

defines a bounded C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on X whose generator $-A$ can be considered as $-\psi(A)$, thus we will write $\psi(A)$ instead of A (see the next subsection for more on that). The C_0 -semigroup $(e^{-t\psi(A)})_{t \geq 0}$ is called subordinated to the C_0 -semigroup $(e^{-tA})_{t \geq 0}$ via the subordinator $(\mu_t)_{t \geq 0}$ (or the corresponding Bernstein function ψ).

Despite the construction of subordination is very natural and appears often in various contexts, some of its permanence properties have not been made precise so far. In this note, we show the permanence of semigroup holomorphicity of under subordination. In particular, we present a positive answer to the following open question posed in [12, p. 63], see also [3]: suppose that $-A$ generates a bounded holomorphic C_0 -semigroup on a

Banach space X and ψ is a Bernstein function. Does $-\psi(A)$ also generate a (bounded) holomorphic C_0 -semigroup ?

A partial answer to a strengthened version of this question was given in [3, Proposition 7.4]: for any Bernstein function ψ the operator $-\psi(A)$ generates a sectorially bounded holomorphic C_0 -semigroup of angle at least θ if $-A$ generates a sectorially bounded holomorphic C_0 -semigroup of angle θ greater than $\pi/4$. Moreover, it was proved in [3, Theorem 6.1 and Remark 6.2] that the above claim is true with no restrictions on $\theta \in (0, \pi/2]$ if a Bernstein function ψ is, in addition, *complete*. It was asked in [3] whether this additional assumption can, in fact, be removed.

If X is a uniformly convex Banach space, e.g. if X is a Hilbert space, then a positive answer to Kishimoto-Robinson's question was obtained in [14, Theorem 1] using Kato-Pazy's criteria for semigroup holomorphicity.

Recently, based on the machinery of functional calculi, positive answers to both questions in their full generality, were provided in [8]. In particular, it was proved in [8] that if $-A$ generates a sectorially bounded holomorphic C_0 -semigroup of angle θ , then for any Bernstein function ψ the operator $-\psi(A)$ also generate a sectorially bounded holomorphic C_0 -semigroup of angle at least θ .

The aim of this note is to present an alternative and comparatively simple argument providing positive answers to the questions from [12] and [3] apart from the permanence of holomorphy angles property. (This property requires additional arguments going beyond the scope of the paper, see [8] for its proof.) Our approach has merits of being self-contained, transparent and much less technical in a sense of using only elementary properties of functional calculi theory.

The proof arises as a byproduct of estimates for $\|\psi(A)e^{-t\varphi(A)}\|$, $t > 0$, where ψ, φ are Bernstein functions, satisfying appropriate conditions. In turn such estimates appeared to be crucial in putting approximation theory of operator semigroups into the framework of Bernstein functions of semigroup generators, see [7]. In fact, the techniques developed in [7] is basic in this paper.

It is not clear whether the permanence of semigroup holomorphy sectors can be proved by the methods of present note. See however [2] where still another, direct approach to subordination was worked out in details.

2. PRELIMINARIES

2.1. Function theory. Let us recall some basic facts on completely monotone and Bernstein functions from [19] relevant for the following.

First, note that by Bernstein's theorem [19, Theorem 1.4] a real-valued function $g \in C^\infty(0, \infty)$ is completely monotone if and only if it is the Laplace transform of a (necessarily unique) positive Laplace-transformable Radon

measure ν on $[0, \infty)$:

$$(2.1) \quad g(\tau) = \widehat{\nu}(\tau) = \int_0^\infty e^{-\tau s} \nu(ds) \quad \text{for all } \tau > 0.$$

In particular, (2.1) implies that a completely monotone function extends holomorphically to the open right half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. The set of complete monotone functions will be denoted by \mathcal{CM} , and the set of bounded complete monotone functions will be denoted by \mathcal{BCM} . The standard examples of completely monotone functions include $e^{-t\tau}$, $\tau^{-\alpha}$, for fixed $t > 0$ and $\alpha \in [0, 1]$, and $(\log(1 + \tau))^{-1}$.

Bernstein functions constitute a class “dual” in a sense to the class of completely monotone functions. A representation similar in a sense to (2.1) holds also for Bernstein functions. Indeed, by [19, Thm. 3.2], a function ψ is a Bernstein function if and only if there exist $a, b \geq 0$ and a positive Radon measure γ on $(0, \infty)$ satisfying

$$\int_{0+}^\infty \frac{s}{1+s} \gamma(ds) < \infty$$

such that

$$(2.2) \quad \psi(\tau) = a + b\tau + \int_{0+}^\infty (1 - e^{-s\tau}) \gamma(ds), \quad \tau > 0.$$

The formula (2.2) is called the Lévy-Khintchine representation of ψ . The triple (a, b, γ) is uniquely determined by ψ and is called the Lévy-Khintchine triple. Thus we will write occasionally $\psi \sim (a, b, \gamma)$. Note that a Bernstein function $\psi \sim (a, b, \gamma)$ is increasing, and it satisfies

$$a = \psi(0+) \quad \text{and} \quad b = \lim_{t \rightarrow \infty} \frac{\psi(t)}{t}.$$

Moreover, by (2.2), ψ extends holomorphically to \mathbb{C}_+ and continuously to the closure $\overline{\mathbb{C}}_+$. The Bernstein function ψ is bounded if and only if $b = 0$ and $\gamma((0, \infty)) < \infty$, see [19, Corollary 3.7].

In the sequel, we will denote the set of Bernstein functions by \mathcal{BF} . As examples of Bernstein functions we mention $1 - e^{-t\tau}$, τ^α , for fixed $t > 0$ and $\alpha \in [0, 1]$, and $\log(1 + \tau)$.

Now we introduce a functional J which will be an important tool in getting operator norm estimates for the products of functions of a negative semigroup generator A .

For $g \in \mathcal{CM}$ and $\psi \in \mathcal{BF}$ let us define

$$(2.3) \quad J[g, \psi] := \int_0^\infty g(s) \psi'(s) ds.$$

Note that J is well-defined if we allow $J[g, \psi]$ to be ∞ .

The following choice of g and ψ will be of particular importance. Observe that if $t > 0$ is fixed, φ is a Bernstein function, and $g = e^{-t\varphi}$ then $g \in \mathcal{BCM}$

by Theorem 1.1 and

$$(2.4) \quad J[e^{-t\varphi}, \psi] = \int_0^\infty e^{-t\varphi(s)} \psi'(s) ds.$$

Let us note several conditions on g and ψ guaranteeing that $J[g, \psi]$ is finite.

Example 2.1. a) Let $g \in \mathcal{CM}$ and $\psi \in \mathcal{BF}$. If there exists a continuous function $q : (0, \infty) \mapsto (0, \infty)$ such that

$$(2.5) \quad \int_0^\infty q(s) ds < \infty, \quad \text{and} \quad g(s) \leq q(\psi(s)), \quad s > 0,$$

then

$$J[g, \psi] \leq \int_0^\infty q(\psi(s)) \psi'(s) ds = \int_{\psi(0)}^{\psi(\infty)} q(s) ds \leq \int_0^\infty q(s) ds < \infty.$$

On the other hand, if $g \in \mathcal{CM}$, $\psi \in \mathcal{BF}$ and $J[g, \psi] < \infty$, then

$$g(\tau) = q(\psi(\tau)), \quad z > 0, \quad q(s) := g(\psi^{-1}(s)), \quad s \in (\psi(0), \psi(\infty)),$$

and

$$\begin{aligned} \int_{\psi(0)}^{\psi(\infty)} q(s) ds &= \int_{\psi(0)}^{\psi(\infty)} g(\psi^{-1}(s)) dt \\ &= \int_0^\infty g(\psi^{-1}(\psi(s))) d\psi(s) = \int_0^\infty g(s) \psi'(s) ds < \infty. \end{aligned}$$

Thus, (2.5) is also necessary (in a sense described above) for $J[g, \psi] < \infty$.

b) Let $g \in \mathcal{BCM}$ be such that $g(0) \leq 1$ and $g(\infty) = 0$, and let $\psi \in \mathcal{BF}$. Suppose that there exists a continuous function $f : (0, 1) \mapsto (0, \infty)$ such that

$$(2.6) \quad \int_0^1 f(s) ds < \infty, \quad \text{and} \quad \psi(s) \leq f(g(s)), \quad s > 0.$$

Then

$$(2.7) \quad J[g, \psi] \leq \int_0^1 f(s) ds.$$

Indeed, note that $g'(s) < 0$, $s < 0$. Then, by (2.6), for all $\epsilon > 0$ and $\tau > 1$,

$$\begin{aligned} (2.8) \quad \int_\epsilon^\tau g(s) \psi'(s) ds &= g(\tau) \psi(\tau) - g(\epsilon) \psi(\epsilon) - \int_\epsilon^\tau g'(s) \psi(s) ds \\ &\leq g(\tau) f(g(\tau)) - \int_\epsilon^\tau g'(s) f(g(s)) ds \\ &= g(\tau) f(g(\tau)) + \int_{g(\tau)}^{g(\epsilon)} f(s) ds \\ &\leq g(\tau) f(g(\tau)) + \int_0^1 f(s) ds. \end{aligned}$$

Note that $g(\tau)$ decrease to zero monotonically as $\tau \rightarrow \infty$. Since $f \in L^1(0, 1)$ there exists $(\tau_k)_{k \geq 1} \subset (1, \infty)$ such that

$$\lim_{k \rightarrow \infty} \tau_k = \infty, \quad \text{and} \quad \lim_{k \rightarrow \infty} g(\tau_k) f(g(\tau_k)) = 0.$$

Since g and ψ' are positive, setting $\tau = \tau_k, k \in \mathbb{N}$, in (2.8) and letting $k \rightarrow \infty$ and $\epsilon \rightarrow 0$, we obtain (2.7).

We proceed with several estimates for $J[g, \psi]$, where g is of the form $e^{-t\varphi}, t > 0$, for a Bernstein function φ . They will be important for exploring holomorphicity of $(e^{-t\varphi(A)})_{t \geq 0}$ in the next section.

Example 2.2. a) For any $\psi \in \mathcal{BF}$, we have

$$\begin{aligned} (2.9) \quad J[e^{-t\psi}, \psi] &= \int_0^\infty e^{-t\psi(s)} \psi'(s) ds \\ &= t^{-1} [e^{-t\psi(0)} - e^{-t\psi(\infty)}] \leq t^{-1}, \quad t > 0. \end{aligned}$$

b) If $\psi \in \mathcal{BF}$ and $\varphi_\alpha(\tau) := \tau^\alpha, \alpha \in (0, 1]$, then using monotonicity of ψ and the fact that

$$(2.10) \quad \psi(c\tau) \leq c\psi(\tau), \quad \tau \geq 0, \quad c \geq 1,$$

see e.g. [11, p. 205], it follows that

$$\begin{aligned} (2.11) \quad J[e^{-t\varphi_\alpha}, \psi] + \psi(0) &= t\alpha \int_0^\infty e^{-ts^\alpha} s^{\alpha-1} \psi(s) ds \\ &= \alpha \int_0^\infty e^{-s^\alpha} s^{\alpha-1} \psi(s/t^{1/\alpha}) ds \\ &\leq \psi(1/t^\alpha) \int_0^\infty e^{-s} \max\{1, s^{1/\alpha}\} ds \\ &\leq \left(1 + \frac{1}{\alpha e}\right) \psi(1/t^\alpha), \quad t > 0. \end{aligned}$$

Let now $\psi \sim (a, b, \gamma)$ and $\alpha = 1$ so that $\varphi_1(\tau) = \tau$. Then using (2.2), the inequality

$$\frac{s}{t+s} = \frac{s/t}{1+s/t} \leq 1 - e^{-s/t}, \quad s, t > 0,$$

and Fubini's theorem, we infer that

$$\begin{aligned} (2.12) \quad J[e^{-t\varphi_1}, \psi] &= \int_0^\infty e^{-tz} \psi'(s) ds = \frac{b}{t} + \int_{0+}^\infty \frac{s}{t+s} \gamma(ds) \\ &\leq \frac{b}{t} + \int_{0+}^\infty (1 - e^{-s/t}) \gamma(ds) = \psi(1/t) - \psi(0) \\ &\leq \psi(1/t), \quad t > 0. \end{aligned}$$

The following estimate for J generalizes the one in a).

c) Let ψ be a bounded Bernstein function satisfying

$$(2.13) \quad \psi(0) = 0, \quad \psi'(0+) < \infty,$$

and let φ be a Bernstein function. Then,

$$J[e^{-t\varphi}, \psi] = \int_0^\infty e^{-t\varphi(s)} \psi'(s) ds \leq \psi(\infty), \quad t > 0.$$

On the other hand, if we are interested in asymptotics of $J[e^{-t\varphi}, \psi]$ for big t and $\varphi \not\equiv \text{const}$, then a better estimate is available. Since

$$\varphi(\tau) = \int_0^\tau \varphi'(s) ds + \varphi(0) \geq \varphi'(1)\tau, \quad \tau \in (0, 1),$$

it follows that

$$\begin{aligned} J[e^{-t\varphi}, \psi] &= \int_0^1 e^{-t\varphi(s)} \psi'(s) ds + \int_1^\infty e^{-t\varphi(s)} \psi'(s) ds \\ &\leq \psi'(0) \int_0^1 e^{-t\varphi'(1)s} ds + e^{-t\varphi(1)} \int_1^\infty \psi'(s) ds \\ &\leq \left[\frac{\psi'(0)}{\varphi'(1)} + \frac{\psi(\infty) - \psi(1)}{\varphi(1)} \right] \frac{1}{t}, \quad t > 0. \end{aligned}$$

We finish this subsection with several estimates shading a light on interplay between the functional $J[g, \psi]$ and the product $g \cdot \psi$. They will be needed as an illustration of our main statement.

The following estimate is well-known for so-called complete Bernstein functions. However, it seems, it has not been noted for the whole class of Bernstein functions. In the proof, we use an idea from the proof of [4, Theorem 4].

Proposition 2.1. Let $\psi \in \mathcal{BF}$. Then

$$(2.14) \quad |\psi(z)| \leq 2\sigma^{-1}\varphi(|z|), \quad \operatorname{Re} z \geq 0, \quad \sigma = 1 - e^{-1}.$$

Proof. Recall that

$$|1 - e^{-z}| \leq \min(|z|, 2) \leq 2\min(|z|, 1), \quad \operatorname{Re} z \geq 0,$$

and

$$1 - e^{-s} \geq \sigma \min(s, 1), \quad s \geq 0, \quad \sigma = 1 - e^{-1},$$

see [11, Lemma 2.1.2]. Therefore,

$$(2.15) \quad |1 - e^{-z}| \leq 2\sigma^{-1}(1 - e^{-|z|}), \quad \operatorname{Re} z \geq 0.$$

Let $\psi \in \mathcal{BF}$ be given by (2.2). Then, using (2.15) and noting that $1 < 2\sigma^{-1}$, we obtain

$$\begin{aligned} |\psi(z)| &\leq a + b|z| + \int_{0+}^\infty |1 - e^{-sz}| \gamma(ds) \\ &\leq a + b|z| + 2\sigma^{-1} \int_{0+}^\infty (1 - e^{-|z|s}) \gamma(ds) \\ &\leq 2\sigma^{-1}\psi(|z|), \quad \operatorname{Re} z \geq 0. \end{aligned}$$

□

In the following result, we show that for $g \in \mathcal{BCM}$ and $\psi \in \mathcal{BF}$ the assumption $J[g, \psi] < \infty$ implies that $g \cdot \psi$ is bounded in any sector

$$\Sigma_\beta := \{z \in \mathbb{C} : |\arg z| < \beta\}, \quad \beta \in (0, \pi/2).$$

Corollary 2.3. *Let $\psi \in \mathcal{BF}$. Then the following statements hold.*

(i) *For every $g \in \mathcal{CM}$ and every $\beta \in (0, \pi/2)$,*

$$(2.16) \quad |g(z)\psi(z)| \leq \frac{2}{\sigma \cos \gamma} g(|z| \cos \beta) \psi(|z| \cos \beta), \quad z \in \Sigma_\beta.$$

(ii) *Let $g \in \mathcal{BCM}$ and $J[g, \psi] < \infty$. Then for every $\beta \in (0, \pi/2)$,*

$$(2.17) \quad |g(z)\psi(z)| \leq \frac{2}{\sigma \cos \beta} \{g(0+)\psi(0) + J[g, \psi]\}, \quad z \in \Sigma_\beta.$$

Proof. To prove (i) suppose that g is given by (2.1) and $z \in \Sigma_\beta$. Then

$$|g(z)| \leq \int_0^\infty e^{-s \operatorname{Re} z} \nu(ds) \leq \int_0^\infty e^{-s|z| \cos \beta} \nu(ds) = g(|z| \cos \beta).$$

Using Proposition 2.1 and the inequality (2.10), we then obtain

$$|g(z)\psi(z)| \leq 2\sigma^{-1} g(|z| \cos \beta) \psi(|z|) \leq \frac{2}{\sigma \cos \beta} g(|z| \cos \beta) \psi(|z| \cos \beta),$$

and the proof is complete.

If $J[g, \psi] < \infty$, then, since g is decreasing, for every $\tau > 0$,

$$\begin{aligned} g(\tau)\psi(\tau) &= g(0+)\psi(0) + \int_0^\tau [g'(s)\psi(s) + g(s)\psi'(s)] ds \\ &\leq g(0+)\psi(0) + \int_0^\tau g(s)\psi'(s) ds \leq g(0+)\psi(0) + J[g, \psi]. \end{aligned}$$

Hence, by (i),

$$|g(z)\varphi(z)| \leq \frac{2}{\sigma \cos \beta} \{g(0+)\phi(0) + J[g, \psi]\}, \quad z \in \Sigma_\beta,$$

so that (ii) holds. \square

2.2. Functional calculus and holomorphic semigroups. In this subsection we will set up the extended Hille-Phillips functional calculus. The calculus will enable us to define Bernstein functions of a negative semigroup generator and to establish some of their basic properties including operator counterparts of the formulas (1.1) and (2.2). As we will see below, the formulas remain essentially the same upon replacement an independent variable by an operator A .

Let $M_b(\mathbb{R}_+)$ be a Banach algebra of bounded Radon measures on $\mathbb{R}_+ := [0, \infty)$ with the standard, total variation norm $\|\mu\|_{M_b(\mathbb{R}_+)} := |\mu|(\mathbb{R}_+)$. Note that

$$A_+^1(\mathbb{C}_+) := \{\widehat{\mu} : \mu \in M_b(\mathbb{R}_+)\}$$

is also a commutative Banach algebra with pointwise multiplication and with the norm inherited from $A_+^1(\mathbb{C}_+)$:

$$(2.18) \quad \|\hat{\mu}\|_{A_+^1(\mathbb{C}_+)} := \|\mu\|_{M_b(\mathbb{R}_+)}.$$

Let $-A$ be the generator of a bounded C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on X . Define an algebra homomorphism $\Phi : A_+^1(\mathbb{C}_+) \mapsto \mathcal{L}(X)$ by the formula

$$\Phi(\hat{\mu})x := \int_0^\infty e^{-sA}x \mu(ds), \quad x \in X.$$

Since

$$(2.19) \quad \|\Phi(\hat{\mu})\| \leq \sup_{t \geq 0} \|e^{-tA}\| \|\mu\|(\mathbb{R}_+),$$

Φ is clearly continuous. The homomorphism Φ is called the *Hille-Phillips* (HP-) functional calculus for A . If $g \in A_+^1(\mathbb{C}_+)$ so that $g = \hat{\mu}$ for $\mu \in M_b(\mathbb{R}_+)$, we then put

$$g(A) = \Phi(\hat{\mu}).$$

Basic properties of the Hille-Phillips functional calculus can be found in [10, Chapter XV] and in [9, Chapter 3.3]. It is crucial to note that if $g \in \mathcal{BCM}$, then $g \in A_+^1(\mathbb{C}_+)$ by Fatou's theorem, so that $g(A)$ is defined in the HP-calculus and $g(A) \in \mathcal{L}(X)$.

Let now $O(\mathbb{C}_+)$ be an algebra of functions holomorphic in \mathbb{C}_+ . Denote by $A_{+,r}^1(\mathbb{C}_+)$ the set of $f \in O(\mathbb{C}_+)$ such that there exists $e \in A_+^1(\mathbb{C}_+)$ with $ef \in A_+^1(\mathbb{C}_+)$ and the operator $e(A)$ is injective. Then for any $f \in A_{+,r}^1(\mathbb{C}_+)$ one defines $f(A)$ as

$$(2.20) \quad f(A) := (e(A))^{-1}ef(A).$$

The above definition does not depend on the choice of a regularizer e , and thus the mapping $f \rightarrow f(A)$ is well-defined. We will call this mapping the *extended Hille-Phillips calculus* for A .

The extended HP-calculus satisfies, in particular, the following, natural sum and product rules, see e.g. [9, Chapter 1].

Proposition 2.4. *Let f and g belong to $A_{+,r}^1(\mathbb{C}_+)$, and let $-A$ be the generator of a bounded C_0 -semigroup. Then*

- (i) $f(A)g(A) \subset (fg)(A)$;
- (ii) $f(A) + g(A) \subset (f+g)(A)$;

If $g(A)$ is bounded then the inclusions above are, in fact, equalities.

Recall that, as it was shown in [6, Lemma 2.5], Bernstein functions are regularisable by $e(z) = 1/(1+z)$, that is $e\psi \in A_+^1(\mathbb{C})$ for every Bernstein function ψ , and then, in particular, by the HP-calculus,

$$(2.21) \quad [\psi(z)(1+z)^{-1}](A) \in \mathcal{L}(X).$$

Thus, according to (2.20), for any $\psi \in \mathcal{BF}$,

$$(2.22) \quad \psi(A) = (1+A)[\psi(z)(1+z)^{-1}](A).$$

While Bernstein functions can formally be defined in the extended HP -calculus by (2.20), this definition can hardly be used for practical purposes. However, following analogy to the scalar-valued case, one can derive representations for operator Bernstein functions similar to (1.1) and (2.2), see e.g. [6, Corollary 2.6] and [19, Proposition 2.1 and Theorem 12.6].

Theorem 2.5. *Let $-A$ generate a bounded C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on X , and let ψ be a Bernstein function with the corresponding Lévy-Hintchine triple (a, b, γ) . Then the following statements hold.*

(i) *For every $x \in \text{dom}(A)$,*

$$(2.23) \quad \psi(A)x = ax + bAx + \int_{0+}^{\infty} (1 - e^{-sA})x \gamma(ds),$$

where the integral is understood as a Bochner integral. Moreover, $\text{dom}(A)$ is core for $\psi(A)$.

(ii) *The operator $-\psi(A)$ generates a bounded C_0 -semigroup $(e^{-t\psi(A)})_{t \geq 0}$ on X given by*

$$(2.24) \quad e^{-t\psi(A)} := \int_0^{\infty} e^{-sA} \mu_t(ds), \quad t \geq 0,$$

where $(\mu_t)_{t \geq 0}$ is a vaguely continuous convolution semigroup of sub-probability measures on $[0, \infty)$ corresponding to ψ by (1.1).

Thus, the operator Bernstein function $\psi(A)$ can be recovered from its restriction to $\text{dom}(A)$ by means of (2.23). Moreover, $-\psi(A)$ generates a bounded C_0 -semigroup if $-A$ does, and this fact motivates further study of the permanence properties for the mapping $-A \rightarrow -\psi(A)$, e.g. preservation of the class of generators of holomorphic semigroups on X .

It will be crucial to note that subordination does not increase the norm. Indeed, as an immediate consequence of Theorem 2.5, (ii), one obtains

$$(2.25) \quad \sup_{t \geq 0} \|e^{-t\psi(A)}\| \leq \sup_{\geq 0} \|e^{-tA}\|.$$

While the relations (2.23) and (2.24) hold for any bounded C_0 -semigroup, in this note we will concentrate on bounded C_0 -semigroups which are, in addition, holomorphic. Recall that a C_0 -semigroup $(e^{-tA})_{t \geq 0}$ is said to be holomorphic if it extends holomorphically to a sector Σ_β for some $\beta \in (0, \frac{\pi}{2}]$ which is bounded on $\Sigma_\theta \cap \{z \in \mathbb{C} : |z| \leq 1\}$ for any $\theta \in (0, \beta)$. If $e^{-\cdot A}$ is bounded in Σ_θ whenever $0 < \theta < \beta$, then $(e^{-tA})_{t \geq 0}$ is said to be a sectorially bounded holomorphic semigroup of angle β .

It is well-known that sectorially bounded holomorphic semigroups can be described by means of their asymptotics on the real axis. Namely, $-A$ is the generator of a sectorially bounded holomorphic C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space X if and only if $e^{-tA}(X) \subset \text{dom}(A)$ for every $t > 0$, and $\sup_{t \geq 0} \|e^{-tA}\|$ and $\sup_{t > 0} \|tAe^{-tA}\|$ are finite, see e.g. [5, Theorem 4.6].

It is often useful to omit the assumption of sectorial boundedness and to consider C_0 -semigroups bounded on \mathbb{R}_+ and having a holomorphic extension

to a sector around the real axis. This situation can also be characterized in terms of behavior of $(e^{-tA})_{t \geq 0}$ on the positive half-axis.

By a classical Yosida's theorem, a C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on X is holomorphic if and only if

$$(2.26) \quad e^{-tA}(X) \subset \text{dom}(A), \quad t > 0, \text{ and } \limsup_{t \rightarrow 0} \|tAe^{-tA}\| < \infty.$$

Since it is no easy to find this statement in the literature, we sketch its proof below. Note that by [1, Proposition 3.7.2 b)] a C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on X is holomorphic if and only if there exists $a > 0$ such that $(e^{-t(A+a)})_{t \geq 0}$ is a sectorially bounded holomorphic C_0 -semigroup. Then, by [5, Theorem 4.6] mentioned above, the latter property is equivalent to $e^{-tA}(X) \subset \text{dom}(A)$ for every $t > 0$, and

$$(2.27) \quad \sup_{t > 0} (e^{-at}\|e^{-tA}\| + \|te^{-at}Ae^{-tA}\|) < \infty.$$

Thus, in particular, (2.26) holds. Conversely, if (2.26) is true, then (2.27) is satisfied for certain $a > 0$, and the sectorial boundedness of $(e^{-t(A+a)})_{t \geq 0}$ yields the holomorphicity of $(e^{-tA})_{t \geq 0}$. (Concerning Yosida's theorem and its proof see also [20] and [13, Remark, p. 332].)

Note that if $(e^{-tA})_{t \geq 0}$ is holomorphic and bounded, then for all $\delta > 0$ and $t > \delta$,

$$\|Ae^{-tA}\| \leq \left(\sup_{t \geq 0} \|e^{-tA}\| \right) \sup_{t \in (\delta/2, \delta)} \|Ae^{-tA}\|.$$

In other words, if $(e^{-tA})_{t \geq 0}$ is bounded, then the Yosida condition (2.26) can be given the equivalent form

$$(2.28) \quad \|Ae^{-tA}\| \leq c_0 + \frac{c_1}{t}, \quad t > 0,$$

with some constants $c_0 \geq 0$ and $c_1 > 0$ which will be crucial in the estimates below. Thus, if $(e^{-tA})_{t \geq 0}$ satisfies (2.28), then we say that $(e^{-tA})_{t \geq 0}$ satisfies the *Yosida condition* $Y(c_0, c_1)$ (which is just an explicit form of the classical Yosida condition (2.26) above).

It will be convenient to rewrite (2.28) in terms of only $(e^{-tA})_{t \geq 0}$. To this aim, we first prove the following simple proposition.

Proposition 2.6. *Let $-A$ be the generator of a bounded C_0 -semigroup on a Banach space X such that*

$$(2.29) \quad \sup_{t \geq 0} \|e^{-tA}\| \leq M.$$

Suppose that $e^{-tA}(X) \subset \text{dom}(A)$, $t > 0$, and there exists an increasing function $r : (0, \infty) \mapsto (0, \infty)$ such that

$$(2.30) \quad \sup_{t > 0} r(t)\|Ae^{-tA}\| \leq 1.$$

Then

$$(2.31) \quad \|(1 - e^{-sA})e^{-tA}\| \leq \frac{4Ms}{2Mr(t) + s}, \quad s, t > 0.$$

Proof. By (2.29), for all $s, t > 0$,

$$\|(1 - e^{-sA})e^{-tA}\| \leq 2M.$$

On the other hand, since

$$(2.32) \quad (1 - e^{-sA})e^{-tA} = \int_t^{t+s} Ae^{-\tau A} d\tau,$$

we infer by (2.30) that

$$\|(1 - e^{-sA})e^{-tA}\| \leq \int_t^{t+s} \frac{d\tau}{r(\tau)} \leq \frac{s}{r(t)}, \quad s, t > 0.$$

Then, since

$$\min\{a, b\} \leq \frac{2ab}{a+b}, \quad a, b > 0,$$

it follows that

$$\|(1 - e^{-sA})e^{-tA}\| \leq \min\{2M, s/r(t)\} \leq \frac{4Ms}{2Mr(t) + s}.$$

□

Now we are ready to recast (2.28) in semigroup terms, and the following corollary of Proposition 2.6 is almost immediate.

Corollary 2.7. *Let $-A$ be the generator of a C_0 -semigroup on X satisfying (2.29) and the Yosida condition $Y(c_0, c_1)$. Then*

$$(2.33) \quad \|(1 - e^{-sA})e^{-tA}\| \leq 2s \left\{ \frac{2Mc_0}{1 + c_0s} + \frac{\max(2M, c_1)}{t + s} \right\}, \quad s, t > 0.$$

Conversely, if estimate (2.33) holds, then $(e^{-tA})_{t \geq 0}$ satisfies the Yosida condition $Y(4Mc_0, 2\max(2M, c_1))$.

Proof. By Proposition 2.6 applied to

$$r(t) := \frac{t}{c_0t + c_1}, \quad t > 0,$$

we obtain that

$$\begin{aligned} \|(1 - e^{-sA})e^{-tA}\| &\leq 4Ms \frac{(c_0t + 1)}{2Mt + (c_0t + c_1)s} \\ &= 4Ms \left\{ \frac{c_0t}{2Mt + (c_0t + c_1)s} + \frac{c_1}{2Mt + (c_0t + c_1)s} \right\} \\ &\leq 4Ms \left\{ \frac{c_0}{2M + c_0s} + \frac{c_1}{2Mt + c_1s} \right\} \\ &\leq 2s \left\{ \frac{2Mc_0}{1 + c_0s} + \frac{\max(2M, c_1)}{t + s} \right\}. \end{aligned}$$

If, conversely, (2.33) is true, then dividing both sides of it by s , using (2.32) and passing to the limit as $s \rightarrow 0+$ for a fixed $t > 0$, we get

$$\|Ae^{-tA}\| \leq 4Mc_0 + 2\frac{\max(2M, c_1)}{t},$$

that is $Y(4Mc_0, 2\max(2M, c_1))$ holds. \square

The elementary estimate (2.33) will play a key role in the subsequent arguments.

3. MAIN RESULTS

To obtain a positive answer to Kishimoto-Robinson's question, we need to show that if $(e^{-tA})_{t \geq 0}$ is a bounded C_0 -semigroup satisfying Yosida's condition, then for any Bernstein function ψ one has $e^{-t\psi(A)}(X) \subset \text{dom}(\psi(A))$, $t > 0$, and the function $t \mapsto \|t\psi(A)e^{-t\psi(A)}\|$ is bounded in an appropriate neighborhood of zero. This will be derived as a simple consequence of the following operator norm estimate for $\psi(A)g(A)$ where $\psi \in \mathcal{BF}$ and $g \in \mathcal{BCM}$. In a different context, a related estimate was obtained in [16, Theorem 1].

For the rest of the paper, if $(e^{-tA})_{t \geq 0}$ is a bounded C_0 -semigroup on a Banach space X then we let

$$M(A) := \sup_{t \geq 0} \|e^{-tA}\|.$$

Theorem 3.1. *Let $\psi \in \mathcal{BF}$ and $g \in \mathcal{BCM}$ be such that $J[g, \psi] < \infty$. Let $-A$ be the generator of a bounded C_0 -semigroup satisfying the Yosida condition $Y(c_0, c_1)$. Then $\psi(A)g(A) \in \mathcal{L}(X)$ and*

$$(3.1) \quad \begin{aligned} \|\psi(A)g(A)\| &\leq \psi(0)\|g(A)\| \\ &+ 2\max(M(A), c_1)J[g, \psi] + 4M(A)g(0+)C[c_0; \psi], \end{aligned}$$

where

$$(3.2) \quad C[c_0; \psi] := \int_0^\infty e^{-s/c_0} \psi'(s) ds, \quad c_0 > 0, \quad C[0; \psi] := 0.$$

Proof. By assumption and Bernstein's theorem, there exists a finite Radon measure ν on $[0, \infty)$ such that

$$(3.3) \quad g(s) = \int_0^\infty e^{-\tau s} \nu(d\tau), \quad s > 0, \quad g(0+) = \nu([0, \infty)) < \infty.$$

Let $\varphi \sim (a, b, \gamma)$ so that the representation (2.2) holds. Then (3.2) takes the form

$$C[c_0; \psi] = bc_0 + \int_{0+}^\infty \frac{c_0 s}{1 + c_0 s} \gamma(ds).$$

Note that it suffices to prove (3.1) for a Bernstein function ψ with $a = \psi(0) = 0$.

Suppose first that $a = b = 0$ in (2.2). Let $x \in \text{dom}(A)$ be fixed. Then, by (2.21) and (2.4),

$$g(A)x \in \text{dom}(A) \subset \text{dom} \psi(A).$$

Hence, by Fubini's theorem, we have

$$\begin{aligned}\psi(A)g(A)x = g(A)\psi(A)x &= \int_0^\infty e^{-\tau A} \nu(d\tau) \int_{0+}^\infty [1 - e^{-sA}] x \gamma(ds) \\ &= \int_0^\infty \int_{0+}^\infty [1 - e^{-sA}] e^{-\tau A} x \gamma(ds) \nu(d\tau).\end{aligned}$$

Using (2.33) and (3.3), from here it follows that

$$\begin{aligned}(3.4) \quad & \|\psi(A)g(A)x\| \\ & \leq 2\|x\| \int_0^\infty \int_{0+}^\infty \left\{ \frac{2M(A)c_0s}{1+c_0s} + \frac{\max(2M, c_1)s}{\tau+s} \right\} \gamma(ds) \nu(d\tau) \\ & = 2\|x\| \left\{ g(0)C[c_0; \psi] + \max(2M(A), c_1) \int_0^\infty \int_{0+}^\infty \frac{s}{\tau+s} \gamma(ds) \nu(d\tau) \right\}.\end{aligned}$$

Again, by applying Fubini's theorem twice, we obtain that (as in (2.12))

$$\int_0^\infty e^{-\tau t} \psi'(t) dt = \int_{0+}^\infty \frac{s \gamma(ds)}{s + \tau}, \quad \tau > 0.$$

and

$$\begin{aligned}(3.5) \quad & \int_0^\infty \int_{0+}^\infty \frac{s \gamma(ds)}{s + \tau} \nu(d\tau) = \int_0^\infty \int_0^\infty e^{-\tau t} \psi'(t) dt \nu(d\tau) \\ & = \int_0^\infty \int_0^\infty e^{-\tau t} \nu(d\tau) \psi'(t) dt \\ & = \int_0^\infty g(t) \psi'(t) dt \\ & = J[g, \psi].\end{aligned}$$

So, (3.4) yields

$$(3.6) \quad \|\psi(A)g(A)x\| \leq 2\|x\| \{ \max(2M(A), c_1) J[g, \psi] + 2M(A)g(0)C[c_0; \psi] \}.$$

From (3.6), since $\psi(A)g(A)$ is closed as product of closed and bounded operators and $\text{dom}(A)$ is dense in X , we conclude that

$$(3.7) \quad \text{ran}(g(A)) \subset \text{dom}(\psi(A)),$$

and (3.1) holds. This finishes the proof in the case $a = b = 0$.

Let now $a = 0$ and $b > 0$. Arguing as above, if $x \in \text{dom}(A)$ is fixed, then

$$\begin{aligned}\psi(A)g(A)x &= g(A)\psi(A)x \\ &= b \int_0^\infty A e^{-\tau A} x \nu(d\tau) + \int_0^\infty \int_{0+}^\infty [1 - e^{-sA}] e^{-\tau A} x \gamma(ds) \nu(d\tau).\end{aligned}$$

Note that $\psi'(s) \geq b$, $s > 0$, and

$$\begin{aligned}\int_0^\infty \tau^{-1} \nu(d\tau) &= \int_0^\infty g(s) ds \\ &\leq b^{-1} \int_0^\infty g(s) \psi'(s) ds = b^{-1} J[g, \psi] < \infty.\end{aligned}$$

Therefore,

$$(3.8) \quad \|Ag(A)x\| \leq \int_0^\infty \|Ae^{-\tau A}x\| \nu(d\tau) \leq \|x\| \int_0^\infty (c_0 + c_1/\tau) \nu(d\tau).$$

Now using (3.5) for a Bernstein function $\psi(t) - bt$, and taking into account (3.8), we obtain that

$$\begin{aligned} \|\psi(A)g(A)x\| &\leq b\|x\| \int_0^\infty (c_0 + c_1\tau^{-1}) \nu(d\tau) \\ &+ 2\|x\| \int_0^\infty \int_{0+}^\infty \left\{ \frac{2M(A)c_0s}{1+c_0s} + \frac{\max(2M(A), c_1)s}{\tau+s} \right\} \gamma(ds) \nu(d\tau) \\ &\leq g(0+)bc_0\|x\| + \|x\|b \int_0^\infty g(s) ds \\ &+ 4M(A)g(0+)\|x\| \int_{0+}^\infty \frac{c_0s}{1+c_0s} \gamma(ds) \\ &+ 2\max(M(A), c_1)\|x\| \int_0^\infty g(s)[\psi'(s) - b] ds \\ &\leq 4M(A)g(0+)\|x\| \left\{ bc_0 + \int_{0+}^\infty \frac{c_0s}{1+c_0s} \gamma(ds) \right\} \\ &+ 2\max(2M(A), c_1)\|x\| \int_0^\infty g(s)\psi'(s) ds \\ &= 2\|x\| \{ \max(2M(A), c_1)J[g, \psi] + 2M(A)g(0+)C[c_0; \psi] \}. \end{aligned}$$

Since the operator $\psi(A)g(A)$ is closed and $\text{dom}(A)$ is dense, the last inequality implies (3.7) and (3.1). \square

Remark 3.2. The assumption $J[g, \psi] < \infty$ is not necessary to ensure the boundedness of $\psi(A)g(A)$. To see this, it is enough to consider a Bernstein function $\psi(\tau) = \tau + 1$ and a bounded completely monotone function $g(\tau) = 1/(\tau + 1)$. However, the assumption implies the boundedness of $\psi \cdot g$ in any sector Σ_β with $\beta \in (0, \pi/2)$, see Corollary 2.3. If $-A$ generates a sectorially bounded holomorphic C_0 -semigroup and admits, in addition, a bounded H^∞ -calculus on a sector Σ_θ , the boundedness of $\psi \cdot g$ in Σ_β , $\beta > \theta$, implies also the boundedness of $\psi(A)g(A)$.

For a choice of g as $e^{-t\varphi}$, where φ is a Bernstein function, Theorem 3.1 yields immediately the following corollaries.

Corollary 3.3. *Let ψ and φ be Bernstein functions such that $J[e^{-t\varphi}, \psi] < \infty$ for every $t > 0$. Let $-A$ be the generator of a bounded C_0 -semigroup on X satisfying the Yosida condition $Y(c_0, c_1)$. Then for every $t > 0$,*

$$\begin{aligned} \|\psi(A)e^{-t\varphi(A)}\| &\leq \psi(0)\|e^{-t\varphi(A)}\| \\ &+ 2\max(2M(A), c_1)J[e^{-t\varphi}, \psi] + 4M(A)e^{-t\varphi(0)}C[c_0, \psi]. \end{aligned}$$

Corollary 3.4. *Let ψ be a Bernstein function and let $-A$ be the generator of a bounded C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on X satisfying the Yosida condition $Y(c_0, c_1)$. Then for every $t > 0$,*

$$(3.9) \quad \|\psi(A)e^{-tA}\| \leq 2\max(2M(A), c_1)\psi(1/t) + 4M(A)C[c_0; \psi].$$

In particular, if $-A$ generates a sectorially bounded holomorphic C_0 -semigroup, then

$$(3.10) \quad \|\psi(A)e^{-tA}\| \leq 2\max(2M(A), c_1)\psi(1/t), \quad t > 0.$$

Proof. By (2.12) and Corollary 3.3 applied to a Bernstein function $\varphi_1(\tau) = \tau$,

$$\begin{aligned} \|\psi(A)e^{-tA}\| &\leq \psi(0)M(A) + 2\max(2M(A), c_1)J[e^{-t\varphi_1}, \psi] + 4M(A)C[c_0; \psi] \\ &\leq 2\max(2M(A), c_1)\{J[e^{-t\varphi_1}, \psi] + \psi(0)\} + 4M(A)C[c_0; \psi] \\ &\leq 2\max(2M(A), c_1)\psi(1/t) + 4M(A)C[c_0; \psi]. \end{aligned}$$

□

As we explained in the beginning of this section, Corollary 3.3 leads to a positive answer to Kishimoto-Robinson's question which is contained in the next statement. Incidentally, it also partially answers the question from [3] and shows that Bernstein functions map the class of generators of sectorially bounded holomorphic C_0 -semigroups into itself. The statement was proved in [8] by a different technique.

Corollary 3.5. *Let ψ be a Bernstein function and let $-A$ be the generator of a bounded C_0 -semigroup satisfying the Yosida condition $Y(c_0, c_1)$. Then $-\psi(A)$ generates a bounded C_0 -semigroup on X satisfying the following Yosida condition:*

$$(3.11) \quad \|\psi(A)e^{-t\psi(A)}\| \leq M(A)(\psi(0) + 4)C[c_0; \psi]e^{-t\psi(0)} + 2\max(2M(A), c_1)t^{-1}$$

for every $t > 0$. If $-A$ generates a sectorially bounded C_0 -semigroup on X , then the same is true for $-\psi(A)$.

Proof. Note that $\psi = \psi(0) + \psi_0$, $\psi_0 \in \mathcal{BF}$, and then

$$(3.12) \quad \|e^{-t\psi(A)}\| \leq e^{-\psi(0)t} \|e^{-t\psi_0(A)}\| \leq M(A), \quad t > 0.$$

Now Corollary 3.3 and Example 2.2, a) yield (3.11). If $(e^{-tA})_{t \geq 0}$ is sectorially bounded, then $c_0 = 0$ and, by definition, $C[c_0; \psi] = 0$ as well. In this case, (3.11) implies that $t\psi(A)e^{-t\psi(A)}$ is bounded on $(0, \infty)$. Since $(e^{-t\psi(A)})_{t \geq 0}$ is bounded, it is moreover sectorially bounded. □

Next we turn to other applications of Theorem 3.1 arising in a general framework for approximation theory of operator semigroups developed in [7]. Note that Corollary 3.3 and Example 2.2, c) imply directly the next statement (cf. [7, Theorem 6.8]).

Theorem 3.6. *Let ψ be a bounded Bernstein function satisfying (2.13), and let $\varphi \not\equiv \text{const}$ be a Bernstein function. Let $-A$ be the generator of a sectorially bounded holomorphic C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on X . Then*

$$(3.13) \quad \sup_{t > 0} \|t\psi(A)e^{-t\varphi(A)}\| \leq 2 \max(2M(A), c_1) \left[\frac{\psi'(0)}{\varphi'(1)} + \frac{\psi(\infty) - \psi(1)}{\varphi(1)} \right].$$

The following corollary of Theorem 3.6 was obtained in [7, Corollary 6.9].

Corollary 3.7. *Let φ be a Bernstein function such that*

$$(3.14) \quad \varphi'(0+) = 1, \quad |\varphi''(0+)| < \infty.$$

Let $-A$ be the generator of a sectorially bounded holomorphic C_0 -semigroup $(e^{-tA})_{t \geq 0}$ on X . Then

$$\|(1 - \varphi'(A))e^{-t\varphi(A)}\| \leq \frac{2 \max(2M(A), c_1)}{t} \left[\frac{|\varphi''(0+)|}{\varphi'(1)} + \frac{\varphi'(1)}{\varphi(1)} \right],$$

for all $t > 0$.

Proof. Note that by (3.14) the Bernstein function $\psi(\tau) = 1 - \varphi'(\tau)$, $\tau > 0$, is bounded and satisfies (2.13). Applying Theorem 3.6 to a Bernstein function φ and a bounded Bernstein function ψ and taking into account the relations $\psi'(0+) = -\varphi''(0+) = |\varphi''(0+)|$ and

$$\psi(\infty) - \psi(1) = \varphi'(1) - \varphi'(\infty) \leq \varphi'(1),$$

we get the assertion. \square

Remark 3.8. Note that in [7, Theorem 6.8] the second term $\frac{\psi(\infty) - \psi(1)}{\varphi(1)}$ in the right hand of (3.13) has a wrong form $\psi(1)/\varphi(1)$ due to incorrect evaluation of $\|\psi'\|_{L^1([a, \infty))} = \int_a^\infty \psi'(s) ds$ in the last line of the proof. Thus [7, Eq. (6.12)] should take a form of (3.13). However, [7, Corollary 6.9] (i.e. Corollary 3.7 here) which was a base for subsequent estimates in [7, Section 6] remains unchanged.

We finish with relating our estimates to the following generalization of the moment inequality for generators of bounded C_0 -semigroups given in [19, Corollary 12.18]. As proved in [19], if $-A$ is the generator of a bounded C_0 -semigroup on X and $\psi \in \mathcal{BF}$, then

$$(3.15) \quad \|\psi(A)x\| \leq \frac{2e}{e-1} M(A) \psi \left(\frac{\|Ax\|}{2\|x\|} \right), \quad x \neq 0, \quad x \in \text{dom}(A).$$

If $\psi(\tau) = \tau^\alpha$, $\alpha \in (0, 1)$, then (3.15) reduces to the classical moment inequality for fractional powers of A . Using our technique, we obtain the following corollary of (3.15).

Corollary 3.9. *Let $-A$ be the generator of a bounded C_0 -semigroup such that*

$$(3.16) \quad \|tAe^{-tA}\| \leq M_a, \quad t \in (0, a], \quad a \leq \infty,$$

and $\psi \in \mathcal{BF}$. Then

$$(3.17) \quad \|\psi(A)e^{-tA}\| \leq \frac{e}{e-1}M(A) \max\{2M(A), M_a\} \psi(1/t), \quad t \in (0, a].$$

Proof. Setting in (3.15) $x = e^{-tA}y$, $y \in X$, $t \in (0, a]$ and using (3.16) and (2.10), we obtain that

$$\begin{aligned} \|\psi(A)e^{-tA}y\| &\leq \frac{2e}{e-1}M(A)\|e^{-tA}y\|\psi\left(\frac{M_a\|y\|}{2t\|e^{-tA}y\|}\right) \\ &\leq \frac{2e}{e-1}M(A)\|e^{-tA}y\|\max\left\{1, \frac{M_a\|y\|}{2\|e^{-tA}y\|}\right\}\psi(1/t) \\ &= \frac{e}{e-1}M(A)\max\{2\|e^{-tA}y\|, M_a\|y\|\}\psi(1/2t) \\ &\leq \frac{e}{e-1}M(A)\max\{2M(A), M_a\}\psi(1/t)\|y\|, \end{aligned}$$

that is (3.17) holds. \square

As an illustration of Corollary 3.9, note that if $\psi(\tau) = \log(1 + \tau)$ then Corollary 3.9 yields the estimate

$$\sup_{t \in (0, 1/e]} \frac{\|\log(1 + A)e^{-tA}\|}{\log(1/t)} < \infty.$$

proved originally in [17, Proposition 2.7].

Finally, we note that it is possible to develop an approach to the permanence problems from [12] and [3] different from the ones in [8] and in the present note. This approach based on direct resolvent estimates for Bernstein functions of semigroup generators is worked out in [2]. While it allows one to get sharp estimates for subordinated semigroups (and their holomorphy sectors), it is much more involved than the arguments in this article.

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